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Elementary proof of $\varepsilon \to +0$ limit of renormalised Feynman amplitudes: II. Theories involving zero-mass particles†

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Abstract. An elementary proof of the distributional $\varepsilon \to +0$ limit of renormalised Feynman amplitudes in momentum space, smeared with Schwartz functions and absolutely convergent for $\varepsilon > 0$, is given for theories involving rigorously *zero-mass* particles. The proof uses only direct and simple methods and establishes the convergence of such amplitudes in Minkowski space.

1. Introduction

An elementary proof of $\varepsilon \to +0$ limit of renormalised Feynman amplitudes has been given recently (Manoukian 1983) for theories with strictly massive particles using only direct and simple methods. For earlier treatments of the problem one may refer to for example, Hepp (1966), Hahn and Zimmermann (1968), Zimmermann (1968), Lowenstein and Speer (1976). The estimates in our earlier proof rely, in an obvious manner, on the existence of a smallest non-zero mass particle in the theory in question, and hence breaks down for theories containing, as a subset of their masses, particles with rigorously zero masses. We were able to extend our elementary proof to such general cases by using direct and simple methods and, in particular, we have avoided altogether the use of existing sophisticated mathematical results in distribution theory involved with the so-called resolution of singularities (Lowenstein and Speer 1976) which are very complex in nature. We prove the distributional $\varepsilon \to +0$ limit of renormalised Feynman amplitudes in momentum space, smeared with Schwartz functions and absolutely convergent for $\varepsilon > 0$, for theories involving rigorously zero-mass particles. This establishes the convergence of such amplitudes in Minkowski space. Needless to say, the existence of field theory models with zero-mass particles which are relevant to the real world makes the proof of the theorem of great importance.

2. Proof of the $\varepsilon \to +0$ limit

A renormalised Feynman amplitude, in momentum space, has the familiar form

$$F_\varepsilon(P, \mu) = \int_{\mathbf{R}^{4n}} \mathrm{d}K\, A(P, K, \mu, \varepsilon) \prod_{l=1}^{L} D_l^{-1}, \qquad \varepsilon > 0 \tag{1}$$

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where

$$D_l = [Q_l^2 + \mu_l^2 - i\varepsilon(\boldsymbol{Q}_l^2 + \mu_l^2)], \qquad \mu_l \geq 0 \tag{2}$$

$Q_l = \Sigma_{i=1}^m a_{li}p_i + \Sigma_{i=1}^n b_{li}k_i$, A is a polynomial in its arguments, and for those $\mu_j > 0$ it may also be, in general, a polynomial in these μ_j^{-1} as well.

To prove the theorem we only require that for any Schwartz function $f(P) \in \mathscr{S}(\mathbb{R}^{4m})$

$$T_\varepsilon(f) = \int_{\mathbb{R}^{4m}} \mathrm{d}P f(P) \int_{\mathbb{R}^{4n}} \mathrm{d}K\, A(P, K, \mu, \varepsilon) \prod_{l=1}^{L} D_l^{-1}, \tag{3}$$

is absolutely convergent *for* $\varepsilon > 0$. We then prove that

$$\lim_{\varepsilon \to +0} \int_{\mathbb{R}^{4m}} \mathrm{d}P f(P) \int_{\mathbb{R}^{4n}} \mathrm{d}K\, A(P, K, \mu, \varepsilon) \prod_{l=1}^{L} D_l^{-1} \tag{4}$$

exists.

Let $A(P, K, \mu, \varepsilon) = \Sigma_a \varepsilon^a A_a(P, K, \mu)$ and hence in an obvious notation $T_\varepsilon(f) = \Sigma_a \varepsilon^a T_\varepsilon^a(f)$. By introducing Feynman parameters we may write (Hahn and Zimmermann 1968, Zimmermann 1968, Lowenstein and Speer 1976, Manoukian 1983, 1984)

$$T_\varepsilon^a(f) = \int_{\mathbb{R}^{4m}} \mathrm{d}P f(P) \int_D \mathrm{d}\alpha\, N^a(\alpha, P, \mu, \varepsilon)[G_\varepsilon(\alpha, P, \mu)]^{-t} \tag{5}$$

where

$$G_\varepsilon(\alpha, P, \mu) = pUp + M^2 - i\varepsilon(\boldsymbol{p}\cdot U\boldsymbol{p} + M^2) \tag{6}$$

$$M^2 = \sum_{l=1}^{L} \alpha_l \mu_l^2 \tag{7}$$

and the matrix U is rational in α and continuous almost everywhere in $D = \{\alpha = (\alpha_1, \ldots, \alpha_L),\ \alpha_i \geq 0,\ \Sigma_{i=1}^L \alpha_i = 1\}$, and may be extended to a continuous function in α everywhere in D (Hahn and Zimmermann 1968, Zimmermann 1968). $N^a(\alpha, P, \mu, \varepsilon)$ is rational in α, and is a polynomial in its remaining arguments, and for those $\mu_j > 0$ it may also be, in general, a polynomial in these μ_j^{-1} as well; t is some positive integer.

We make use of the following useful identities (Manoukian 1983, 1984)

$$\begin{aligned}[G_\varepsilon(\alpha, P, \mu)]^{-t} = &-(\boldsymbol{p}\cdot U\boldsymbol{p} + M^2)^2 t(t+1) \int_\varepsilon^1 \mathrm{d}\lambda_1 \int_{\lambda_1}^1 \mathrm{d}\lambda [G_\lambda(\alpha, P, \mu)]^{-t-2} \\ &+ (\boldsymbol{p}\cdot U\boldsymbol{p} + M^2) it(\varepsilon - 1)[G_1(\alpha, P, \mu)]^{-t-1} + [G_1(\alpha, P, \mu)]^{-t}\end{aligned} \tag{8}$$

$$[G_\lambda(\alpha, P, \mu)]^{-t-2} = \frac{(-\frac{1}{2})^{t+1}}{(t+1)!}\left((p^0Up^0)^{-1} \sum_{i=1}^{m} p_i^0 \frac{\partial}{\partial p_i^0}\right)^{t+1} [G_\lambda(\alpha, P, \mu)]^{-1}. \tag{9}$$

When the second and the third terms on the right-hand side of (8) are replaced in turn for $[G_\varepsilon(\alpha, P, \mu)]^{-t}$ in (5) and the resulting expressions are denoted, respectively, by $T_\varepsilon^{a2}(f)$, $T_\varepsilon^{a3}(f)$, then one readily obtains (Manoukian 1984, see also lemma A1, (i), (ii), (iv) in the appendix)

$$|T_\varepsilon^{ai}(f)| \leq C_i \int \mathrm{d}P|f(P)| \int \mathrm{d}K|A_a(P, K, \mu)| \prod_{l=1}^{L} D_{lE}^{-1} < \infty, \qquad i = 2, 3, \tag{10}$$

where $D_{lE} = \boldsymbol{Q}_l^2 + Q_l^{02} + \mu_l^2$. The existence of the $\varepsilon \to +0$ limit of $T_\varepsilon^{ai}(f)$, for $i = 2, 3$, then follows by an elementary application of the Lebesgue dominated convergence theorem. The difficulty in proving (4) is not due to $T_\varepsilon^{a2}(f)$, $T_\varepsilon^{a3}(f)$ but to the term $T_\varepsilon^{a1}(f)$ which is obtained by substituting the first term on the right-hand side of (8) for $[G_\varepsilon(\alpha, P, \mu)]^{-t}$ in (5). Now we focus our attention to $T_\varepsilon^{a1}(f)$. To this end we introduce a $\mathscr{C}^\infty$ function $0 \leq \chi(x) \leq 1$ (Lowenstein and Speer 1976, Manoukian 1983, 1984) defined by $\chi(x)=0$ for $x<\frac{1}{3}$, and $\chi(x)=1$ for $x>\frac{2}{3}$. We set $x \equiv p^0Up^0/(\boldsymbol{p}\cdot U\boldsymbol{p}+M^2)$. We then write $T_\varepsilon^{a1}(f)=T_\varepsilon^{a1}(f)_1+T_\varepsilon^{a1}(f)_2$, where

$$T_\varepsilon^{a1}(f)_1 = -t(t+1)\int_{\mathbb{R}^{4m}} \mathrm{d}P f(P)\int_D \mathrm{d}\alpha \int_\varepsilon^1 \mathrm{d}\lambda_1 \int_{\lambda_1}^1 \mathrm{d}\lambda N^a(\alpha, P, \mu, \varepsilon)[1-\chi(x)] \times(\boldsymbol{p}\cdot U\boldsymbol{p}+M^2)^2[G_\lambda(\alpha, P, \mu)]^{-t-2}, \tag{11}$$

$$T_\varepsilon^{a1}(f)_2 = -t(t+1)\int_{\mathbb{R}^{4m}} \mathrm{d}P f(P)\int_D \mathrm{d}\alpha \int_\varepsilon^1 \mathrm{d}\lambda_1 \int_{\lambda_1}^1 \mathrm{d}\lambda\, N^a(\alpha, P, \mu, \varepsilon)\chi(x) \times(\boldsymbol{p}\cdot U\boldsymbol{p}+M^2)^2[G_\lambda(\alpha, P, \mu)]^{-t-2}. \tag{12}$$

Due to the presence of the function $[1-\chi(x)]$ in (11), we effectively have in (11) $p^0Up^0 \leq 2(pUp+M^2)$ and therefore $[p_EUp_E+M^2] \leq 5[pUp+M^2]$. Hence for all $\lambda \geq 0$ we have *in* (11)

$$\begin{aligned}[G_\lambda(\alpha, P, \mu)]^{-t-2} &\leq [pUp+M^2]^{-t-2} \leq C_t[p_EUp_E+M^2]^{-t-2} \\ &\leq C_t[p_EUp_E+M^2]^{-t}(\boldsymbol{p}\cdot U\boldsymbol{p}+M^2)^{-2}.\end{aligned} \tag{13}$$

$$(p_EUp_E \equiv \boldsymbol{p}\cdot U\boldsymbol{p}+p^0Up^0+M^2).$$

Therefore

$$\begin{aligned}|T_\varepsilon^{a1}(f)_1| &\leq C_t'\int_{\mathbb{R}^{4m}} \mathrm{d}P|f(P)|\int_D \mathrm{d}\alpha\int_\varepsilon^1 \mathrm{d}\lambda_1\int_{\lambda_1}^1 \mathrm{d}\lambda|N^a(\alpha, P, \mu, \varepsilon)|[p_EUp_E+M^2]^{-t} \\ &\leq C_t''\int_{\mathbb{R}^{4m}} \mathrm{d}P|f(P)|\int_{\mathbb{R}^{4m}} \mathrm{d}K|A_a(P, K, \mu)|\prod_{l=1}^L D_{lE}^{-1}(<\infty)\end{aligned} \tag{14}$$

for $0 \leq \varepsilon \leq 1$, and where in writing the last inequality we have used lemma A1 (iv). The existence of $\lim_{\varepsilon\to+0} T_\varepsilon^{a1}(f)_1$ then follows from the Lebesgue dominated convergence theorem. Now we turn our attention to $T_\varepsilon^{a1}(f)_2$. Upon writing

$$N^a(\alpha, P, \mu, \varepsilon) = \sum_{b,c} \varepsilon^b p_0^c N_{bc}^a(\alpha, \boldsymbol{P}, \mu), \tag{15}$$

using the identity in (9), integrating over P by parts and using the fact that $f(P)$, together with all of its derivatives, vanish rapidly at infinity, we obtain

$$T_\varepsilon^{a1}(f)_2 = -\sum_{b,c}\varepsilon^b\frac{t(t+1)}{(t+1)!}\left(\frac{1}{2}\right)^{t+1}\int_{\mathbb{R}^{4m}}\mathrm{d}P\int_D \mathrm{d}\alpha\int_\varepsilon^t \mathrm{d}\lambda_1\int_{\lambda_1}^1 \mathrm{d}\lambda\, N_{bc}^a(\alpha, \boldsymbol{P}, \mu) \times[G_\lambda(\alpha, P, \mu)]^{-1}(\boldsymbol{p}\cdot U\boldsymbol{p}+M^2)^2(p^0Up^0)^{-t-1}\sum_i \chi_c^i(p^0, x, \alpha)h_c^i(P), \tag{16}$$

where we remark that χ_c^i may be bounded by a polynomial in p^0 independent of α, and it vanishes for $x<\frac{1}{3}$ due to the property of the function $\chi(x)$; $h_c^i(P) \in \mathscr{S}(\mathbb{R}^{4m})$. Hence we have effectively in (16) the bound

$$|(p^0Up^0)^{-1}(p^0Up^0)^{-t}| \leq C'[\boldsymbol{p}\cdot U\boldsymbol{p}+M^2]^{-1}[p_EUp_E+M^2]^{-t}. \tag{17}$$

Also we may use the bound as in (Manoukian 1984)

$$G_\lambda(\alpha, P, \mu)|^{-1} \leqslant 1/\lambda[\boldsymbol{p}\cdot U\boldsymbol{p}+M^2]. \tag{18}$$

Accordingly,

$$|T_\varepsilon^{a1}(f)_2| \leqslant \sum_{b,c} \varepsilon^b |C_t| \int \mathrm{d}P \sum_i |\tilde{h}_c^i(P)| \left| \int_\varepsilon^1 \mathrm{d}\lambda_1 \int_{\lambda_1}^1 \frac{\mathrm{d}\lambda}{\lambda} \right| \int_D \mathrm{d}\alpha$$
$$\times |N_{bc}^a(\alpha, \boldsymbol{P}, \mu)| [p_E U p_E + M^2]^{-t} < \infty, \tag{19}$$

for all $0 \leqslant \varepsilon \leqslant 1$, by using finally lemma A1 (v). The existence of $\lim_{\varepsilon\to+0} T_\varepsilon^{a1}(f)_2$ then follows from the Lebesgue dominated convergence theorem. This completes the proof of the statement in (4). Explicity we have for $\lim_{\varepsilon\to+0} T_\varepsilon(f) = T_0^0(f)$, where the latter is given by

$$-\int_{\mathbf{R}^{4m}} \mathrm{d}P f(P) \int_D \mathrm{d}\alpha\, N^0(\alpha, P, \mu, 0)(\boldsymbol{p}\cdot U\boldsymbol{p}+M^2) it[G_1(\alpha, P, \mu)]^{-t-1}$$
$$+\int_{\mathbf{R}^{4m}} \mathrm{d}P f(P) \int_D \mathrm{d}\alpha\, N^0(\alpha, P, \mu, 0)[G_1(\alpha, P, \mu)]^{-t}$$
$$-t(t+1) \int_{\mathbf{R}^{4m}} \mathrm{d}P f(P) \int_D \mathrm{d}\alpha \int_0^1 \mathrm{d}\lambda_1 \int_{\lambda_1}^1 \mathrm{d}\lambda\, N^0(\alpha, P, \mu, 0)[1-\chi(x)]$$
$$\times[\boldsymbol{p}\cdot U\boldsymbol{p}+M^2]^2[G_\lambda(\alpha, P, \mu)]^{-t-2}$$
$$-\sum_c \frac{(\frac{1}{2})^t}{(t-1)!} \int_{\mathbf{R}^{4m}} \mathrm{d}P \int_D \mathrm{d}\alpha \int_0^1 \mathrm{d}\lambda_1 \int_{\lambda_1}^1 \mathrm{d}\lambda\, N_{0c}^0(\alpha, P, \mu)[G_\lambda(\alpha, P, \mu)]^{-1}$$
$$\times(\boldsymbol{p}\cdot U\boldsymbol{p}+M^2)^2(p^0 U p^0)^{-t-1} \sum_i \chi_c^i(p^0, x, \alpha) h_c^i(P). \tag{20}$$

Appendix

In this appendix we give an elementary lemma which is useful in the estimates used in the rest of the paper.

Lemma

(i)
$$\int_{\mathbf{R}^{4m}} \mathrm{d}P f(P) \int_{\mathbf{R}^{4m}} \mathrm{d}K\, A(P, K, \mu, \varepsilon) \prod_{l=1}^{L} D_{lE}^{-1}, \tag{A1}$$

is absolutely convergent for $\varepsilon \geqslant 0$.

(ii) Let $A(P, K, \mu, \varepsilon) = \Sigma_a\, \varepsilon^a A_a(P, K, \mu)$, then the expression in (A1) with $A(P, K, \mu, \varepsilon)$ in it replaced by $A_a(P, K, \mu)$ is abolustely convergent for $\varepsilon \geqslant 0$.

(iii) The expression in (3) with $A(P, K, \mu, \varepsilon)$ in it replaced by $A_a(P, K, \mu)$ is absolutely convergent for $\varepsilon > 0$.

(iv)
$$\int_{\mathbf{R}^{4m}} \mathrm{d}P |f(P)| \int_D \mathrm{d}\alpha |N^a(\alpha, P, \mu, \varepsilon)| [p_E U p_E + M^2]^{-t}$$
$$\leqslant C \int_{\mathbf{R}^{4m}} \mathrm{d}P |f(P)| \int_{\mathbf{R}^{4m}} \mathrm{d}K |A_a(P, K, \mu)| \prod_{l=1}^{L} D_{lE}^{-1} < \infty, \qquad \varepsilon \geqslant 0.$$

(v) $$\int_{\mathbb{R}^{4m}} \mathrm{d}P|f(P)| \int_D \mathrm{d}\alpha |N^a_{bc}(\alpha, \boldsymbol{P}, \mu)|[p_E U p_E + M^2]^{-t} < \infty,$$

where $N^a_{bc}(\alpha, \boldsymbol{P}, \mu)$ is defined in (15).

The proof of (i) follows from the estimate (Zimmermann 1968) $|Q^2+\mu^2-\mathrm{i}\varepsilon(\boldsymbol{Q}^2+\mu^2)| \leq C_\varepsilon(Q_E^2+\mu^2)$, $C_\varepsilon > 0$ for $\varepsilon \geq 0$. The proof of (ii) follows from part (i) and the fact that $\varepsilon \geq 0$ is arbitrary. The proof of (iii) follows from (ii) and the estimate (Zimmermann 1968) $(Q_E^2+\mu^2) \leq C'_\varepsilon|Q^2+\mu^2-\mathrm{i}\varepsilon(\boldsymbol{Q}^2+\mu^2)|$, $C'_\varepsilon > 0$, for $\varepsilon > 0$. The proof of (iv) is given in (Manoukian 1984). To prove (v) scale p^0 by a parameter $\lambda > 0$ (Lowenstein and Speer 1976) in (5) in Euclidean space. The resulting expression then becomes

$$\int_{\mathbb{R}^{4m}} \mathrm{d}Pf(\boldsymbol{P}, \lambda p^0) \int_D \mathrm{d}\alpha \sum_{b,c} \varepsilon^b p^{0c}(\lambda)^{g+|c|} N^a_{bc}(\alpha, \boldsymbol{P}, \mu) \times [\boldsymbol{p}\cdot U\boldsymbol{p} + \lambda^2 p^0 U p^0 + M^2]^{-t}, \tag{A2}$$

and is absolutely convergent. Using the elementary inequality

$$[\boldsymbol{p}\cdot U\boldsymbol{p} + \lambda^2 p^0 U p^0 + M^2)/(\boldsymbol{p}\cdot U\boldsymbol{p} + p^0 U p^0 + M^2) \leq 1 + |\lambda^2 - 1|,$$

we obtain from the absolute convergence of the integral in (A2) that

$$\int_{\mathbb{R}^{4m}} \mathrm{d}Pf(\boldsymbol{P}, \lambda P^0) \int_D \mathrm{d}\alpha \sum_{b,c} \varepsilon^b p^{0c}(\lambda)^{g+|c|} N^a_{bc}(\alpha, \boldsymbol{P}, \mu)[p_E U p_E + M^2]^{-t}, \tag{A3}$$

is absolutely convergent. Since the parameters $\lambda > 0$, $\varepsilon \geq 0$ are arbitrary the statement in (v) follows. (Here it is worth noting that due to the property of $f(\boldsymbol{P}, \lambda P^0)$, given any positive, arbitrarily large, integer N we may find a positive constant D, and choose a positive constant d arbitrarily large such that

$$|f(\boldsymbol{P}, \lambda P^0)| \leq D\Big(1 + d\sum_{i=1}^m \boldsymbol{p}_i^2 + \mathrm{d}\lambda^2 \sum_{i=1}^m p_i^{02}\Big)^{-N} \leq D\Big(1 + d\sum_{i=1}^m \boldsymbol{p}_i^2 + \sum_{i=1}^m p_i^{02}\Big)^{-N},$$

with $\lambda^2 \geq d^{-1}$ for the validity of the latter inequality. Equation (A3) is absolutely convergent with $f(\boldsymbol{P}, \lambda P^0)$ in it replaced by $(1 + d\sum_{i=1}^m \boldsymbol{p}_i^2 + \sum_{i=1}^m p_i^{02})^{-N}$ and with N chosen sufficiently large).

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